

T-dual-coordinate dependence makes the effective Kalb-Ramond field nontrivial *

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Abstract

We show that the fact that the string theory is unoriented does not necessarily force the Kalb-Ramond field to vanish. We investigated the theory of the open string propagating in the weakly curved background. The effective Kalb-Ramond field $B_{\mu\nu}^{eff}$, the background field of the effective theory obtained on the solution of the boundary conditions, does not depend on the Ω -even effective coordinate q , but on its T-dual \tilde{q} which is Ω -odd. This brakes the standard proof that the term with $B_{\mu\nu}^{eff}$ should vanish. From the world-sheet equations of motion we identify $B_{\mu\nu}^{eff}$ with the torsion potential.

It is well known that type I superstring theory is unoriented (symmetric under world-sheet parity transformation $\Omega : \sigma \rightarrow -\sigma$). It can be obtained from type IIB superstring theory as an Ω -projection. This procedure eliminates all the states which are odd under Ω -transformation [1]-[4].

Generally, the Ω -odd background fields are absent in the unoriented theories, because they come within terms which are integrated out of the action, as the Ω -odd terms on the symmetric σ interval $[-\pi, \pi]$. This can be proven using the implicit assumption that the background fields depend on the Ω -even coordinates.

We are going to present an exception of this rule, on the example of the bosonic string. We will consider the propagation of the open bosonic string, described by the action [1, 2, 3, 5]

$$S = \kappa \int_{\Sigma} d^2\xi \left[\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x) \right] \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu}, \quad (0.1)$$

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($\varepsilon^{01} = -1$), where integration goes over two-dimensional world-sheet Σ parameterized by the coordinates $\xi^0 = \tau$, $\xi^1 = \sigma$ with $\sigma \in [0, \pi]$. Here $x^\mu(\xi)$, $\mu = 0, 1, \dots, D-1$ are the coordinates of the D-dimensional space-time, and we use the notation $\dot{x} = \frac{\partial x}{\partial \tau}$, $x' = \frac{\partial x}{\partial \sigma}$.

Requirement for the world-sheet conformal invariance on the quantum level, produces the restriction on the background fields which can be represented in a form of the space-time equations of motion. We will consider the following particular solution of these equations, the *weakly curved* background [6, 7, 8, 9]

$$G_{\mu\nu} = \text{const}, \quad B_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho, \quad (0.2)$$

where the parameter $b_{\mu\nu}$ is constant and $B_{\mu\nu\rho}$ is constant and infinitesimally small.

The minimal action principle for the open string produces the equation of motion

$$\ddot{x}^\mu = x''^\mu - 2B_{\nu\rho}^\mu \dot{x}^\nu x'^\rho, \quad (0.3)$$

and the boundary conditions on the string endpoints. For the Neumann boundary conditions we have

$$\gamma_\mu^0 \Big|_{\sigma=0,\pi} = 0, \quad (0.4)$$

where

$$\gamma_\mu^0 \equiv \frac{\delta \mathcal{L}}{\delta x'^\mu} = G_{\mu\nu} x'^\nu - 2B_{\mu\nu} \dot{x}^\nu. \quad (0.5)$$

We treat the boundary conditions as constraints. We demand their time consistency and therefrom we obtain the infinite set of constraints $\gamma_\mu^n \Big|_{\sigma=0,\pi} = 0$, $\gamma_\mu^{n+1} \equiv \dot{\gamma}_\mu^n$, ($n \geq 0$). Applying the procedure developed in Refs.[8, 9], using the equation of motion (0.3), we obtain their explicit form in the leading order in $B_{\mu\nu\rho}$. Instead of working with this infinite set of constraints, we build a σ -dependent constraint at each string end-point. Separating constraint at $\sigma = 0$ in Ω -symmetric and antisymmetric parts $\Gamma_\mu = \Gamma_\mu^S + \Gamma_\mu^A$, after a long calculation described in [8, 9], its compact form has been obtained

$$\begin{aligned} \Gamma_\mu^S(\sigma) &= G_{\mu\nu} \bar{q}^\nu - 2b_{\mu\nu} \dot{q}^\nu - \frac{2}{3}B_{\mu\nu\rho} \left[\dot{q}^\nu q^\rho + \frac{1}{2} \dot{Q}^\nu q'^\rho \right. \\ &\quad \left. + \frac{3}{2} \dot{\bar{q}}^\nu \bar{q}^\rho \right] + 2b_\mu{}^\rho B_{\rho\nu\sigma} \left[q'^\nu \bar{q}^\sigma + \dot{Q}^\nu \dot{\bar{q}}^\sigma \right], \\ \Gamma_\mu^A(\sigma) &= G_{\mu\nu} \dot{\bar{q}}^\nu - 2b_{\mu\nu} q'^\nu - \frac{2}{3}B_{\mu\nu\rho} \left[q'^\nu q^\rho + \frac{1}{2} \dot{Q}^\nu \dot{q}^\rho \right. \\ &\quad \left. + \frac{3}{2} \bar{q}'^\nu \bar{q}^\rho \right] + 2b_\mu{}^\rho B_{\rho\nu\sigma} \frac{\partial}{\partial \sigma} \left[\dot{Q}^\nu \bar{q}^\sigma \right], \end{aligned} \quad (0.6)$$

where we introduced even and odd coordinate variables

$$\begin{aligned} q^\mu(\sigma) &\equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{\mu(2n)} \Big|_{\sigma=0}, \\ \bar{q}^\mu(\sigma) &\equiv \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{\mu(2n+1)} \Big|_{\sigma=0}, \end{aligned} \quad (0.7)$$

and

$$\dot{Q}^\mu(\sigma) = \int_0^\sigma d\eta \dot{q}^\mu(\eta). \quad (0.8)$$

We solve the constraints $\Gamma_\mu^S(\sigma) = 0$, $\Gamma_\mu^A(\sigma) = 0$ by iteration method. The solution, in the zero order in the infinitesimal parameter $B_{\mu\nu\rho}$ is

$$\bar{q}'^\mu = 2b^\mu{}_\nu \dot{q}^\nu, \quad \dot{\bar{q}}^\mu = 2b^\mu{}_\nu q'^\nu. \quad (0.9)$$

In the first order in $B_{\mu\nu\rho}$, it becomes

$$\begin{aligned} \bar{q}'^\mu &= 2B^\mu{}_\nu(q) \dot{q}^\nu - A^\mu{}_\nu(\dot{Q}) q'^\nu, \\ \dot{\bar{q}}^\mu &= 2B^\mu{}_\nu(q) q'^\nu - A^\mu{}_\nu(\dot{Q}) \dot{q}^\nu, \end{aligned} \quad (0.10)$$

where we introduced antisymmetric infinitesimal tensor

$$\begin{aligned} A_{\mu\nu}(\dot{Q}) &= h_{\mu\nu}(\dot{Q}) - 12b_\mu{}^\rho h_{\rho\sigma}(\dot{Q}) b^\sigma{}_\nu \\ &\quad - 12h_{\mu\rho}(b\dot{Q}) b^\rho{}_\nu + 12b_\mu{}^\rho h_{\rho\nu}(b\dot{Q}), \end{aligned} \quad (0.11)$$

with $h_{\mu\nu}(x) \equiv \frac{1}{3} B_{\mu\nu\rho} x^\rho$.

If we extend the σ domain and demand 2π -periodicity of the original variable $x^\mu(\sigma + 2\pi) = x^\mu(\sigma)$, it can be shown that the constraints at $\sigma = 0$ and $\sigma = \pi$ are equivalent. Therefore, the relation (0.10) solves both constraints at $\sigma = 0$ and $\sigma = \pi$.

Substituting the solution into the Lagrangian (0.1) we obtain the effective one

$$\begin{aligned} \mathcal{L}^{eff} &= \frac{\kappa}{2} \dot{q}^\mu G_{\mu\nu}^{eff}(q, \dot{Q}) \dot{q}^\nu - \frac{\kappa}{2} q'^\mu G_{\mu\nu}^{eff}(q, \dot{Q}) q'^\nu \\ &\quad + 2\kappa q'^\mu B_{\mu\nu}^{eff}(q, \dot{Q}) \dot{q}^\nu, \end{aligned} \quad (0.12)$$

where

$$\begin{aligned} G_{\mu\nu}^{eff}(q, \dot{Q}) &= G_{\mu\nu}^E(q + 2b\dot{Q}) + 4[b^2 A(\dot{Q}) - A(\dot{Q}) b^2]_{\mu\nu}, \\ B_{\mu\nu}^{eff}(q, \dot{Q}) &= [h(2b\dot{Q}) + 4bh(2b\dot{Q})b]_{\mu\nu} - B_\mu{}^\rho(q) G_{\rho\nu}^E(q), \end{aligned} \quad (0.13)$$

and

$$G_{\mu\nu}^E(x) \equiv G_{\mu\nu} - 4B_{\mu\rho}(x)(G^{-1})^{\rho\sigma} B_{\sigma\nu}(x) \quad (0.14)$$

is the open string metric.

Because our basic variable $q^\mu(\sigma)$ contains only even powers of σ , it is convenient to regard it as the even function $q^\mu(-\sigma) = q^\mu(\sigma)$ on the interval $\sigma \in [-\pi, \pi]$. Therefore, we will consider the action $S^{eff} = \int d\tau \int_{-\pi}^\pi d\sigma \mathcal{L}^{eff}$, which makes our effective theory unoriented closed string theory. Consequently, the terms of the effective metric which depend

on \dot{Q} and the terms of effective Kalb-Ramond field which depend on q will disappear and we obtain

$$S^{eff} = \kappa \int_{\Sigma_1} d^2\xi \left[\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu}^{eff}(q) + \epsilon^{\alpha\beta} B_{\mu\nu}^{eff}(2b\dot{Q}) \right] \partial_\alpha q^\mu \partial_\beta q^\nu, \quad (0.15)$$

where Σ_1 marks the changed sigma domain $\sigma \in [-\pi, \pi]$. The effective background fields are equal to

$$G_{\mu\nu}^{eff}(q) = G_{\mu\nu}^E(q), \quad B_{\mu\nu}^{eff}(2b\dot{Q}) = -\frac{\kappa}{2} [g\Delta\theta(2b\dot{Q})g]_{\mu\nu}, \quad (0.16)$$

where $\Delta\theta$ is the infinitesimal part of the so called non-commutativity parameter

$$\theta^{\mu\nu} = -\frac{2}{\kappa} [G_E^{-1} B G^{-1}]^{\mu\nu}, \quad (0.17)$$

and $g_{\mu\nu} = G_{\mu\nu}^E(0)$ is the constant part of the effective metric.

There are two unexpected results here. The first one is the appearance of the non-trivial Kalb-Ramond field $B_{\mu\nu}^{eff}$ and the second one is the fact that it does not depend on the coordinate q^μ but on $\dot{Q}^\mu(\sigma) = \int_0^\sigma d\eta \dot{q}^\mu(\eta)$. We are going to offer an explanation and the interpretation of these results.

It is well known that the theory of the unoriented closed string (which is just our effective theory) should not contain the Kalb-Ramond field. Let us first present the standard reasons, for this statement [2, 10]. The effective Kalb-Ramond field appears in the effective action within the term $B_{\mu\nu}^{eff} \dot{q}^\mu \dot{q}^\nu$. So, as \dot{q}^μ is Ω -even and q'^μ Ω -odd variable, if the Kalb-Ramond field depends on the Ω -even variable, this term does not contribute, because it disappears after integration over the symmetric interval $[-\pi, \pi]$.

What is different in our case? It is the fact that the effective Kalb-Ramond field does not depend on the effective coordinate q^μ (Ω -even) but on the integral of the τ -derivative of the effective coordinate $\dot{Q}_\mu(\sigma) = \int_0^\sigma d\eta \dot{q}_\mu(\eta)$ (Ω -odd). Since, $B_{\mu\nu}^{eff}(2b\dot{Q})$ is linear in \dot{Q}^μ it means that the effective Kalb-Ramond field is odd under σ -parity transformation $\Omega B_{\mu\nu}^{eff}[2b\dot{Q}(\sigma)] = -B_{\mu\nu}^{eff}[2b\dot{Q}(\sigma)]$. This minus sign changes the situation, because the term in the action with the effective Kalb-Ramond field becomes Ω -even, and this fact allows its survival.

What can the interpretation of \dot{Q}^μ be? First, note that \dot{Q}^μ appears as an argument of $B_{\mu\nu}^{eff}$ only, which is the infinitesimal of the first order. So, it is enough to consider \dot{Q}^μ up to the zero order. The zero order equation of motion for q^μ is just $\partial_+ \partial_- q^\mu = 0$, and consequently the solution has the form $q^\mu(\sigma) = f^\mu(\sigma^+) + g^\mu(\sigma^-)$, with $\sigma^\pm = \tau \pm \sigma$. The Ω -evenness of the variable q^μ , $q^\mu(-\sigma) = q^\mu(\sigma)$, implies $f(\sigma) = g(\sigma)$ and we obtain

$$q^\mu(\sigma) = f^\mu(\sigma^+) + f^\mu(\sigma^-). \quad (0.18)$$

From the properties $\partial_{\pm} f^{\mu}(\sigma^{\mp}) = 0$, we have $\dot{f}^{\mu}(\sigma^{\pm}) = \pm f'^{\mu}(\sigma^{\pm})$. Therefore, $\dot{q}^{\mu}(\sigma) = f'^{\mu}(\sigma^{+}) - f'^{\mu}(\sigma^{-})$, and consequently

$$\dot{Q}^{\mu}(\tau, \sigma) = f^{\mu}(\sigma^{+}) - f^{\mu}(\sigma^{-}) \equiv \tilde{q}^{\mu}(\tau, \sigma), \quad (0.19)$$

where $\tilde{q}^{\mu}(\tau, \sigma)$ is T-dual mapping of the effective coordinate $q^{\mu}(\tau, \sigma)$ (see for example (17.76) of Ref. [2] or eq. (6.17) of Ref.[3]). Note that T-dual coordinate \tilde{q} is Ω -odd variable, because $\tilde{q}(-\sigma) = -\tilde{q}(\sigma)$. So, in the effective theory with dynamical variable q^{μ} , the effective metric depends on the coordinate q^{μ} and the effective Kalb-Ramond field on its T-dual \tilde{q}^{μ} .

The question arises, why do the background fields depend on the different arguments? To understand this, it is enough to consider the zero order solution of the boundary conditions, because the arguments of the background fields appear only in the infinitesimally small terms. Let us rewrite the solution of the constraints (0.9), in the form

$$x^{\mu}(\sigma) = q^{\mu}(\sigma) + 2(G^{-1}b)^{\mu}_{\nu} \tilde{q}^{\nu}. \quad (0.20)$$

The effective metric depends on the first (Ω -even) part of x^{μ} and the effective Kalb-Ramond field on the second (Ω -odd) part of x^{μ} . So, we can formally rewrite the effective action (0.15) in the form

$$S^{eff} = \kappa \int_{\Sigma_1} d^2\xi \left[\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu}^{eff}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}^{eff}(x) \right] \partial_{\alpha} q^{\mu} \partial_{\beta} q^{\nu}, \quad (0.21)$$

because $\eta^{\alpha\beta} [G_{\mu\nu}^{eff}(2b\tilde{q}) - G_{\mu\nu}^{eff}(0)] \partial_{\alpha} q^{\mu} \partial_{\beta} q^{\nu}$ and $\epsilon^{\alpha\beta} B_{\mu\nu}^{eff}(q) \partial_{\alpha} q^{\mu} \partial_{\beta} q^{\nu}$ do not contribute being the Ω -odd terms. Therefore, formally, both background fields depend on the same argument, the solution for the initial coordinate x^{μ} expressed as in (0.20), in terms of the effective one q^{μ} and its T-dual \tilde{q}^{μ} .

The appearance of the T-dual effective coordinate $\tilde{q}^{\mu} = \dot{Q}^{\mu}$ in the Lagrangian (0.21) suggests that we have obtained the nonlocal theory. It is interesting to find the equation of motion of the theory of this type, when $G_{\mu\nu}^{eff}$ is arbitrary function of q^{μ} and $B_{\mu\nu}^{eff}$ consists of the odd powers in Ω -odd variable $\bar{q} = 2b\tilde{q}$.

Variation with respect to q^{μ} produces world-sheet equation of motion

$$\eta^{\alpha\beta} {}^0D_{\beta} \partial_{\alpha} q^{\mu} = 2\kappa \theta_0^{\mu\nu} \epsilon^{\alpha\beta} \int_0^{\sigma} d\eta \frac{\partial}{\partial \tau} \left[\frac{\partial B_{\rho\sigma}^{eff}(\bar{q})}{\partial \bar{q}^{\nu}} \partial_{\alpha} q^{\rho} \partial_{\beta} q^{\sigma} \right], \quad (0.22)$$

where ${}^0D_{\alpha} V^{\mu} = \partial_{\alpha} q^{\nu 0} D_{\nu} V^{\mu}$ is the covariant derivative in the world-sheet direction. ${}^0D_{\mu}$ is generalized covariant derivative ${}^0D_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + {}^0\Gamma_{\rho\mu}^{\nu} V^{\rho}$, with the generalized connection

$${}^0\Gamma_{\rho\sigma}^{\mu} = (\Gamma_{eff})_{\rho\sigma}^{\mu} + \frac{1}{2} {}^0K_{\rho\sigma}^{\mu}. \quad (0.23)$$

It consists of the Christoffel connection $(\Gamma_{eff})_{\rho\sigma}^{\mu}$ and contorsion ${}^0K_{\mu\rho\sigma} = \frac{1}{2}{}^0T_{\{\sigma\mu\rho\}}$ in terms of torsion

$${}^0T_{\mu\rho\sigma} = 4b_{\rho}^{\nu} \frac{\partial}{\partial \bar{q}^{\nu}} B_{\mu\sigma}^{eff}. \quad (0.24)$$

Here $\{\mu\nu\rho\} = \nu\rho\mu + \rho\mu\nu - \mu\nu\rho$ is Schouten bracket.

Therefore, the Kalb-Ramond field is related to the torsion potential. This is in accordance with the usual interpretation of the Kalb-Ramond field $B_{\mu\nu}$ in the low energy string theory as the non-Rimannian theory [11]. In our case torsion is infinitesimally small constant.

Let us summarize. In our approach the initial theory represents an oriented open string theory. The theory obtained on the solution of the boundary conditions, to which we refer as an effective theory, is an unoriented closed string theory, because it is symmetric under σ -parity transformation, $\Omega : \sigma \rightarrow -\sigma$ and satisfies the boundary condition $q^{\mu}(\sigma = -\pi) = q^{\mu}(\sigma = \pi)$.

The complete transition from the original theory (0.1) to the effective theory (0.15) consists of

1. *the dynamical variable transition*

$$x^{\mu} \rightarrow q^{\mu}, \quad (0.25)$$

2. *the background field transition*

$$G_{\mu\nu} \rightarrow G_{\mu\nu}^{eff}(q), \quad B_{\mu\nu}(x) \rightarrow B_{\mu\nu}^{eff}(2b\tilde{q}). \quad (0.26)$$

The effective action (0.15) describes the propagation of the effective unoriented closed string in the effective background (0.26).

When both background fields of the initial theory, $G_{\mu\nu}$ and $B_{\mu\nu}$, are constant the effective background has constant effective metric $g_{\mu\nu} = G_{\mu\nu}^E(0)$ and zero Kalb-Ramond field. Infinitesimal correction of the initial antisymmetric field $B_{\mu\nu}$, linear in coordinate x^{μ} produces infinitesimal correction of the effective metric linear in effective coordinates q^{μ} and turns the infinitesimal part of the Kalb-Ramond field into the T-dual effective coordinate \tilde{q}^{μ} dependent. This fact makes the term with $B_{\mu\nu}^{eff}$, Ω -even and allows its existence in the unoriented string theory.

The non-commutativity properties of the open string in the weakly curved background are considered in Refs. [8, 9].

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